Western Number Theory Problems, 16 17, & 18 Dec 2018

for distribution prior to 2019 (Asilomar) meeting

Edited by Gerry Myerson based on notes by Kjell Wooding

Summary of earlier meetings & problem sets with old (pre 1984) & new numbering.

1967 Berkeley	1968 Berkeley	1969 Asilomar	
1970 Tucson	1971 Asilomar	1972 Claremont	72:01–72:05
1973 Los Angeles	73:01–73:16	1974 Los Angeles	74:01–74:08
1975 Asilomar	75:01–75:23		
1976 San Diego	1–65 i.e., 76:01–76:65		
1977 Los Angeles	101–148 i.e., 77:01–77:48		
1978 Santa Barbara	151–187 i.e., 78:01–78:37		
1979 Asilomar	201–231 i.e., 79:01–79:31		
1980 Tucson	251–268 i.e., 80:01–80:18		
1981 Santa Barbara	301–328 i.e., 81:01–81:28		
1982 San Diego	351–375 i.e., 82:01–82:25		
1983 Asilomar	401–418 i.e., 83:01–83:18		
1984 Asilomar	84:01–84:27	1985 Asilomar	85:01–85:23
1986 Tucson	86:01–86:31	1987 Asilomar	87:01–87:15
1988 Las Vegas	88:01–88:22	1989 Asilomar	89:01–89:32
1990 Asilomar	90:01–90:19	1991 Asilomar	91:01–91:25
1992 Corvallis	92:01–92:19	1993 Asilomar	93:01–93:32
1994 San Diego	94:01–94:27	1995 Asilomar	95:01–95:19
1996 Las Vegas	96:01–96:18	1997 Asilomar	97:01–97:22
1998 San Francisco	98:01–98:14	1999 Asilomar	99:01–99:12
2000 San Diego	000:01–000:15	2001 Asilomar	001:01–001:23
2002 San Francisco	002:01–002:24	2003 Asilomar	003:01–003:08
2004 Las Vegas	004:01–004:17	2005 Asilomar	005:01–005:12
2006 Ensenada	006:01–006:15	2007 Asilomar	007:01–007:15
2008 Fort Collins	008:01–008:15	2009 Asilomar	009:01–009:20
2010 Orem	010:01–010:12	2011 Asilomar	011.01–011.16
2012 Asilomar	012:01–012:17	2013 Asilomar	013.01–013.13
2014 Pacific Grove	014:01–014:11	2015 Pacific Grove	015:01–015:15
2016 Pacific Grove	016:01–016:14	2017 Pacific Grove	017:01–017:21
2018 Chico	018:01–018:19		

COMMENTS ON ANY PROBLEM WELCOME AT ANY TIME

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Comments on earlier problems

016:04 (Carl Pomerance) Let

$$S = \left\{ n : n \text{ divides } \binom{2n}{n} \right\}$$

This set is infinite, but misses a positive proportion of positive integers. In particular, it misses all those n with a prime factor exceeding $\sqrt{2n}$, a set of density $\log 2$. On the other hand, if $n = pq$ where p and q are prime and $1.5p < q < 2p$, then n is in S. Does S contain a positive proportion of the positive integers? Does S have a density?

Remarks: 1. These numbers are tabulated at http://oeis.org/A014847

2. Pante Stanica conjectures that for $n \geq 3700$ we have

$$\frac{n}{(\log\log n)^3} \leq \#S \leq \frac{n}{(\log\log n)^2}$$

Solution: Kevin Ford and Sergei Konyagin, Divisibility of the central binomial coefficient $\binom{2n}{n}$, https://arxiv.org/abs/1909.03903, show that for every fixed positive integer ℓ, the set of n such that $n^\ell \mid \binom{2n}{n}$ has a positive density c_ℓ. They find c_1 to be approximately 0.114247.

017:15 (Colin Weir) Fix an odd prime p. For how many positive integers $n \leq x$ is the Sylow p-subgroup of $(\mathbf{Z}/n\mathbf{Z})^\times$ cyclic? Sungjin Kim gets

$$C\frac{x\log\log x}{(\log x)^{1/(p-1)}} + \mathrm{O}(x(\log x)^{-1/(p-1)})$$

where $C = C_p$ is given by

$$\frac{1}{p\Gamma((p-2)/(p-1))}\left(\frac{p-1}{p}\prod_{\chi\neq\chi_0}\left(L(1,\chi)\prod_p\left(1-\frac{1}{p}\right)^{-\chi(p)}\left(1-\frac{\chi(p)}{p}\right)\right)\right)^{-\frac{1}{p-1}}$$

Details in separate document from Sungjin. Carl asks how $C = C_p$ grows with p. Perhaps there is a transition in behavior around $(\log\log x)^{1+\epsilon}$.

Problems proposed 16 to 18 December 2018

018:01 (Enrique Treviño) For what k are there k consecutive n-gonal numbers that add up to an n-gonal number?

Remark: Enrique has some results for the triangular case. See Pollack, Subramaniam, Treviño, Triangular sums of consecutive triangular numbers, http://campus.lakeforest.edu/trevino/triangular.pdf

018:02 (User pi66 on Mathoverflow, via Gerry Myerson) Let $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ be permutations of nonconstant arithmetical progressions of natural numbers. For which n can $(a_1b_1, \ldots, a_nb_n)$ be an arithmetic progression?

Remarks: 1. The citation is https://mathoverflow.net/questions/312896/product-of-arithmetic-progressions

2. The cases $n = 1$ and $n = 2$ are trivial. Examples for the cases $3 \le n \le 6$ are

$(11, 5, 8), (2, 3, 1) \longrightarrow (22, 15, 8)$

$(1, 11, 6, 16), (10, 4, 13, 7) \longrightarrow (10, 44, 78, 112)$

$(8, 6, 4, 7, 5), (4, 9, 19, 14, 24) \longrightarrow (32, 54, 76, 98, 120)$

$(7, 31, 19, 13, 37, 25), (35, 11, 23, 41, 19, 29) \longrightarrow (245, 341, 437, 533, 629, 725)$

3. Yaakov Baruch showed that for each n the question can be reduced to a finite search. He carried out the search for $n = 7$, and found that there are no solutions.

4. User empy2 showed how to streamline Baruch's approach, and reported carrying out the searches for $n = 8$ and $n = 9$, and finding there are no solutions.

018:03 (Gary Walsh) Friedlander and Iwaniec proved that $x^4 + y^2$ is prime infinitely often. Is it a product of two distinct odd primes infinitely often? Can the methods of Friedlander-Iwaniec be used to prove this?

Solution: David Nguyen sketched two solutions to this question. They appear at the end of this report.

018:04 (Carl Pomerance) According to Sheldon Cooper in The Big Bang Theory, 73 is the best number. It is the 21st prime, and $7 \times 3 = 21$—call this the product property. Reversing the digits gives 37, which is the 12th prime, and 12 is the reverse of 21 — call this the mirror property. Chris Spicer (Morningside College) and two undergraduates in Math Horizons proved that 73 is the only prime with the two properties. Pomerance and Spicer proved 73 is the only number, prime or otherwise, with both properties — see Sheldon Primes, https://www.math.dartmouth.edu/˜carlp/sheldon100518.pdf

Now consider just the product property. By the Prime Number Theorem, there are only finitely many primes with this property; indeed, no such prime can exceed 10^{45}. A search found two others besides 73, but the search was not carried out all the way up to 10^{45}. The question is, are there any beyond the three known examples?

Remarks: 1. It is more efficient to search for the product than to search for the primes, since the product must be 7-smooth, and such numbers are uncommon.

2. One can look at the generalization of the problem to bases other than ten.

018:05 (Stephen Glasby, via Tim Trudgian, via Gerry Myerson) Let $\Phi_n(t)$ be the nth cyclotomic polynomial. We define a partial order $\preceq$ on the natural numbers as follows: $m \preceq n$ means $\Phi_m(t) \le \Phi_n(t)$ for all $t \ge 2$. Thus we have, for example,

$$1 \preceq 2 \preceq 6 \preceq 4 \preceq 3 \preceq 10$$

since

$$t - 1 \le t + 1 \le t^2 - t + 1 \le t^2 + 1 \le t^2 + t + 1 \le t^4 - t^3 + t^2 - t + 1$$

for $t \ge 2$. Are the natural numbers totally ordered by this relation?

Remark: There is a tabulation at https://oeis.org/A206225. It has been shown that $\{ 1, 2, \ldots, 20000 \}$ is totally ordered under $\preceq$.

Solution: Carl Pomerance and Simon Rubinstein-Salzedo, Cyclotomic coincidences (preprint available at Carl's webpages), show that this gives a total ordering of the naturals. In outline:

If $f, g \in \mathbf{Z}[t]$, we write $f \prec g$ if $f(t) < g(t)$ for all $t > 2$

Claim: The cyclotomic polynomials are all comparable with each other under $\prec$; i.e they are totally ordered

Clearly f, g are comparable if and only if $f - g$ has no roots > 2. So the claim is implied by the following, and is in fact equivalent to it:

Theorem 1.1: If $m \neq n$ are positive integers and x is a nonzero real number with $\Phi_m(x) = \Phi_n(x)$, then $1/2 < |x| < 2$, except for $\Phi_2(2) = \Phi_6(2)$.

An important citation is: (Bang) for $n \neq 1, 6$ there is a prime p where 2 has order n mod p. Such a prime p has to divide $\Phi_n(2)$.

There are some interesting near-counterexamples, of the form $\Phi_{pq} - \Phi_r$, with p, q, r prime:

$\Phi_{209} - \Phi_{179}$ has a root at 1.99975...

$\Phi_{221} - \Phi_{191}$ has a root at 1.999935...

$\Phi_{527} - \Phi_{479}$ has a root at 1.9999961...

On Dickson's prime k-tuples conjecture, there are examples with a root arbitrarily close to 2.

018:06 (Kjell Wooding) What is the shortest string containing all permutations of a set of n elements? For $n \geq 2$, an upper bound $n! + (n-1)! + (n-2)! + (n-3)! + n - 3$ was found by Greg Egan, and a lower bound is $n! + (n-1)! + (n-2)! + n - 3$. The problem is to improve on these bounds.

Remarks: 1. This is known as the Haruhi problem — see http://mathsci.wikia.com/wiki/The_Haruhi_Problem.

2. Carl Pomerance notes an appearance in Quanta Magazine, https://www.quantamagazine.org/sci-fi-writer-greg-egan-and-anonymous-math-whiz-advance-permutation-problem-20181105

3. A more formal discussion is at https://oeis.org/A180632/a180632.pdf

4. The upper bound is discussed at Greg Egan's site, http://www.gregegan.net/SCIENCE/Superpermutations/Superpermutations.html

018:07 (Carl Pomerance) Are there large gaps between pairs of twin primes? That is, long stretches of consecutive integers not exceeding x with no twin primes? A sieve upper bound gives $c(\log x)^2$. Can this be beaten? Is it true that for any $c > 0$ and $x > x_0(c)$ there are more than $c(\log x)^2$ consecutive integers not exceeding x with no twin primes?

Remarks: 1. Carl reminds us of Hendrik Lenstra's proof that there are infinitely many composites: assume there are finitely many, multiply them together, and *don't* add one.

2. Carl also reminds us of Erdős' proof that there are strings of consecutive composites below x of length

$$\frac{e^\gamma \log x \log\log x \log\log\log\log x}{(\log\log\log x)^2}$$

and of a recent improvement by Ford, Konyagin, Maynard, and Tao; also, of similar work by those four authors and Pomerance improving on $c \log x$ for the longest consecutive sequence of n below x with $n^2 + 1$ composite.

3. There was a question about numerical evidence (Carl wasn't aware of any), and a question about whether sieve methods for showing convergence of the sum of the reciprocals of the twin primes would be useful here.

018:08 (Rachana Madhukara) Faulhaber's formula for $S_p = \sum_{k=1}^{n} k^p = 1^p + 2^p + \cdots + n^p$ can be given as

$$S_p = \frac{n^{p+1}}{p+1} + \frac{1}{2}n^p + \sum_{k=2}^{p} \frac{B_k p!}{k!(p-k+1)!} n^{p-k+1}$$

where the B_k are the Bernoulli numbers. E.g.,

$S_0 = n$,

$S_1 = (1/2)n + (1/2)n^2$,

$S_2 = (1/6)n + (1/2)n^2 + (1/3)n^3$. Entering the coefficients into a lower triangular matrix and inverting produces the binomial coefficients, e.g.,

$$G_7 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 1/6 & 1/2 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 1/4 & 1/2 & 1/4 & 0 & 0 & 0 \\ -1/30 & 0 & 1/3 & 1/2 & 1/5 & 0 & 0 \\ 0 & -1/12 & 0 & 5/12 & 1/2 & 1/6 & 0 \\ 1/42 & 0 & -1/6 & 0 & 1/2 & 1/2 & 1/7 \end{pmatrix},$$

$$G_7^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 3 & 0 & 0 & 0 & 0 \\ -1 & 4 & -6 & 4 & 0 & 0 & 0 \\ 1 & -5 & 10 & -10 & 5 & 0 & 0 \\ -1 & 6 & -15 & 20 & -15 & 6 & 0 \\ 1 & -7 & 21 & -35 & 35 & -21 & 7 \end{pmatrix}$$

(example taken from https://en.wikipedia.org/wiki/Faulhaber%27s_formula).

Are there other polynomials that make interesting patterns like this?

Remarks: 1. The Faulhaber polynomials form a basis for the vector space of all polynomials vanishing at zero, and the matrix is the transformation matrix to the basis $\{ n, n^2, \ldots, \}$. The inverse matrix is the matrix for the change of basis in the other direction. There are other examples where both matrices have entries that are interesting from a number-theoretical or combinatorial point of view. There may be some in Comtet's combinatorics book, and/or in Riordan's.

2. Simon Rubinstein-Alzedo points out that if s is the lower triangular matrix of Stirling numbers of the first kind, then the inverse of this matrix is S, the lower triangular matrix of Stirling numbers of the second kind.

018:09 (V. Dimitrov, via Mike Jacobson) Suppose N is known to be the sum of two B-smooth numbers. How quickly can we find B-smooth numbers a, b such that $N = a + b$? Assume that B is around $(\log N)^2$, so that there are not too many solutions.

Remark: This may be the basis of a cryptographic one-way function.

018:10 (Andrew Hone, via Gerry Myerson) An NSW number (named after Newman, Shanks, and Williams) is a positive integer q such that $q^2 + 1$ is twice a square. These are the numbers $1, 7, 41, \ldots$ of http://oeis.org/A002315, the numerators of the lower principal convergents to $\sqrt{2}$. Hone conjectures

1. infinitely many of these q are prime, and

2. $\log\log(n\text{th NSW prime})$ should be asymptotic to Cn, with $C = (1/2)e^{-\gamma}\log(3+\sqrt{8})$.

Remarks: 1. The first eight NSW primes are listed at http://oeis.org/A088165 Letting a_n be the nth NSW number, with $a_0 = 1$, these primes are a_i for $i = 1, 2, 3, 9, 14, 23, 29, 81$. It says there that the ninth NSW prime "is too large (99 digits) to include in sequence."

2. Carl and Colin asked whether i must be prime for a_i to be prime. That's not the case, but it may be one can prove that if a_i is prime then $2i + 1$ is prime, compare http://oeis.org/A005850

3. The problems are presumably at the same level of difficulty as the analogous problems for Mersenne primes (i.e., hopeless). The conjectured distribution function presumably rests on the same sort of heuristics as inform the analogous estimates for Mersenne primes.

4. David Nguyen proves that the number of primes $p \le x$ such that p^2+1 is a square is $\ll x(\log x)^{-3/2}$, using the square sieve and the Cebotarev Density Theorem. On the Generalized Riemann Hypothesis, he gets $\ll x^{18/19}$. The tools are contained in A. C. Cojocaru, É. Fouvry, M. Ram Murty, The square sieve and the Lang-Trotter conjecture, Canad. J. Math. 57 (2005) 1155–1177.

018:11 (Sungjin Kim) Let $G = \mathrm{SL}_2(\mathbf{Z})$. G acts on the upper halfplane $\mathbf{H}$. Given a real number a, and a positive number d, $0 < d < 1$, the lines through a with slopes d and $-d$ enclose a wedge W in $\mathbf{H}$. Given a point $P = (a_0, b_0)$ in $\mathbf{H}$, we want to estimate the number N of points in the orbit of P under G that lie in the wedge W. When a is rational, we can prove $N \ll_a (1/d)\log(1/d)$. What can be said for irrational a?

018:12 (Enrique Treviño) If we know the order of the natural numbers, and we know the Mobius function $\mu(n)$ for all n, can we factor all n? E.g., $\mu(1) = 1$ and since 1 is the smallest natural it has no prime factors.

$\mu(2) = -1$, so 2 must be the smallest prime.

$\mu(3) = -1$, so 3 must be the second smallest prime.

$\mu(4) = 0$, so 4 must be the square of the smallest prime, $4 = 2^2$.

$\mu(5) = -1$, so 5 must be the third prime.

$\mu(6) = 1$, so 6 must be the product of the two smallest primes, $6 = 2 \times 3$.

$\mu(7) = -1$ can't be a product of three primes, since $2 \times 3 \times 5 > 3 \times 5$ which hasn't yet appeared.

$\mu(8) = \mu(9) = 0$; which one is 2^3, which is 3^2?

D. Flath and A. Zulauf, Does the Möbius function determine multiplicative arithmetic, Amer. Math. Monthly 102 (1995) 354–356, showed that knowing $\mu(m)$ for $1 \le m \le 240$ one can factor all $n \le 74$.

018:13 (Erik Tou) Given primes z_0 and z_1 in $\mathbf{Z}[i]$ (viewed as sitting in the complex plane), if we write N for moduli and θ for arguments of complex numbers, then

$$N = N_0 + \frac{N_1 - N_0}{\theta_1 - \theta_0}(\theta - \theta_0)$$

is an equation for a spiral through z_0 and z_1. How many primes can lie on one such spiral?

Remarks: 1. You can have an infinity of inert primes, by making sure every odd integer lies on the spiral.

2. You can't have more than two split primes, according to a paper by Calcut in the Monthly.

3. What happens if we ask the question for other quadratic fields?

4. David Nguyen suggested that a very recent Friedlander-Iwaniec preprint might help.

5. Bart Goddard noted that if your two primes are conjugate, or associates, then the spiral degenerates to a circle.

018:14 (Kjell Wooding) Are there any paper collections we would like to see preserved?

Remark: Kjell gives links to collections of papers by

Waldschmidt, https://webusers.imj-prg.fr/ michel.waldschmidt/texts.html

Keith Conrad, http://www.math.uconn.edu/ kconrad/blurbs/

and Dujella. https://web.math.pmf.unizg.hr/ duje/

018:15 (Kyle Hammer) $3 \times 24 = 72$ is the reverse of $3 + 24 = 27$. Some pairs (a, b) for which ab is the reverse of $a + b$ are $(2, 2), (9, 9), (3, 24), (2, 47)$. There are more examples, if you permit leading zeros, e.g., $57 \times 13830 = 788310$, $57 + 13830 = 013887$, and, for all n, $2 \times 10^n + 2 \times 10^n$ vs $(2 \times 10^n)^2$. Let

$$R(x) = \#\{\, n \le x : \exists a, b \text{ in } \mathbf{N} \text{ s.t. } a + b = \text{reverse}(ab) = n \,\}$$

What are the asymptotics for $R(x)$?

Remark: "Numbers k such that if $k = ab$, then $a + b = \text{reversal}(k)$ for some integers $a, b > 1$" is tabulated at http://oeis.org/A161791 — leading zeros are permitted.

018:16 (Colin Weir) Are ratios of primes dense in the positive reals?

Solution: Theorem 4 of David Hobby and D. M. Silberger, Quotients of primes, Amer. Math. Monthly, Vol. 100, No. 1 (Jan., 1993), pp. 50-52, answers this question in the affirmative. At https://mathoverflow.net/questions/117191 Keith Conrad gives this simple argument: By the prime number theorem, the nth prime p_n admits the asymptotic estimate $p_n \sim n \log n$. It follows for any real number $x > 0$ that $p_{[nx]}/p_n \to x$ as $n \to \infty$. The question has also come up several times on math.stackexchange.com

018:17 (Gerry Myerson, from Greg Martin, from Joe Silverman) Find a zeta zero with an irrational imaginary part.

Remark: Simon asks whether (nontrivial) zeta zeros are known/believed to be periods.

018:18 (Bart Goddard) 1. Let α be a real irrational. Is it true that for all n there exist odd integers r, s with $|\alpha - (r/s)| < n^{-2}$, $s < n$?

2. Let $\alpha_1, \dots, \alpha_k$ be real irrationals. Is it true that for all n there exist odd integers $r_1, \dots, r_k, s$ with $s < n$ and

$$\left|\alpha_i - \frac{r_i}{s}\right| < n^{-c}$$

where c, which may depend on k, exceeds 1? For example, $c = 1 + (1/k)$.

Remarks: 1. Bart posted the second question to math.stackexchange as question 1895371 in 2016.

2. See Faustin Adiceam, Rational approximation and arithmetic progressions, Int. J. Number Theory 11 (2015) 451–486; "A reasonably complete theory of the approximation

of an irrational by rational fractions whose numerators and denominators lie in prescribed arithmetic progressions is developed in this paper," according to the abstract.

018:19 (Rachana Madhukara) What is the average number of triples (p, q, r) of primes satisfying $n = p^2 + q^2 - r^2$?

Remarks: 1. This appeared in MathOverflow question 318977, since deleted.

2. Simon notes that without the primality condition the number of such triples is infinite for all n.

Here are David Nguyen's sketches of two proofs that $x^2 + y^4$ is a product of two distinct primes infinitely often (**018:03**, Gary Walsh).

Let $a_n = \#\{ n : n = a^2 + b^4, a > 0, b > 0 \}$, and let $\Lambda_2(n) = \sum_{d|n} \mu(d)(\log n/d)^2$. It suffices to obtain an asymptotic estimate

$$\sum_{n \le x, (n,2)=1} \mu(n)^2 a_n \Lambda_2(n) \sim B(x)$$

for some $B(x) \to \infty$ as $x \to \infty$. Applying Möbius inversion, it suffices to show

$$\sum_{n \le x} \mu(n)^2 a_n \Lambda_2(n) \sim B(x)/\log x. \tag{1}$$

First method. Let $A(x) = \sum_{n \le x} a_n$, $A_d(x) = \sum_{n \le x, n \equiv 0 \bmod d} a_n$, let χ be the nontrivial Dirichlet character mod4, and let $g(n)$ be the multiplicative function given for prime p by

$$g(p) = \frac{1}{p} + \left(\frac{\chi(p)}{p}\right)\left(1 - \frac{1}{p}\right)$$

Then it can be verified that

(1) $A(x) \gg A(\sqrt{x})(\log x)^2$,

(2) $A_d(x) \ll g(d)A(x)$ for all $d \le x^{1/3}$,

(3) $\sum_{d \le y} \mu(d)^2 g(d) = c_1 \log y + c_0 + O((\log y)^{-8})$ for some constants $c_1 > 0$ and c_0, and

(4) $\sum_{d \le y} \mu(d) g(d) \ll (\log y)^{-8}$.

Let $r_d(t) = A_d(t) - g(d)A(t)$. Fouvry and Iwaniec [3] prove that for all $\epsilon > 0$ and all $A > 0$ we have

$$\sum_{d \le x^{3/4-\epsilon}} \mu(d)^2 |r_d(t)| \le A(x)(\log x)^{-A}$$

for all $t \le x$.

Let $\gamma(n, C) = \sum_{d|n, d \le C} \mu(d)$. Friedlander and Iwaniec [2] prove for every $A > 0$ and every C, $1 \le C \le x^{1/4+\epsilon}$ we have

$$\sum_m \left| \sum_{N < n \le 2N, mn \le x} \gamma(n, C)\mu(mn)a_{mn} \right| \le A(x)(\log x)^{-A} \tag{2}$$

for every N, $x^{1/4+\epsilon} < N < x^{1/2}(\log x)^{-A}$. Let

$$H = \prod_p (1 - g(p))(1 - 1/p)^{-1}. \tag{3}$$

Then from [2] we have

$$\sum_{n \le x} \mu(n)^2 a_n \Lambda(n) \sim HA(x). \tag{4}$$

Now replace Λ with Λ_2. Following [2], decompose Λ_2 using combinatorial identities, such as the Heath-Brown identity, into a sum of multiple subsums (this part needs some work, as the Heath-Brown identity is for Λ, but one should be able to adapt his proof to Λ_2). Then applying the machinery of [2] should give the asymptotic for (1), which should be larger than that in (4). Working out this asymptotic will be the most laborious part of the solution.

Second method. We need properties H_1 to H_8 of $A(x)$ from Friedlander and Iwaniec [1], and we also need (2). Then Theorem 3 of [1] gives

$$\sum_{n \le x} \mu(n)^2 a_n \Lambda_k(n) \sim kHA(x)(\log x)^{k-1}$$

with H as in (3). Modulo checking that the conditions for Bombieri's asymptotic sieve apply, and letting $k = 2$, we get

$$\sum_{n \le x} \mu(n)^2 a_n \Lambda_2(n) \sim 2HA(x) \log x. \tag{5}$$

This shows $a^2 + b^4$ is a product of at most two distinct primes infinitely often. Since the right side of (5) exceeds that of (4), $a^2 + b^4$ is a product of exactly two distinct odd primes infinitely often.

[1] J. Friedlander and H. Iwaniec, Bombieri's sieve, in B. C. Berndt et al., eds., Analytic Number Theory, Proc. Halberstam Conf. (1996) 411–430.

[2] J. Friedlander and H. Iwaniec, The polynomial $X^2 + Y^4$ captures its primes, Ann. Math. (2) 148(3) (1998) 945–1040.

[3] É. Fouvry and H. Iwaniec, Gaussian primes, Acta Arith. 79(3) (1997) 249–287.