Absolute limit formulas for Hurwitz zeta functions

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The Bernoulli polynomial family of series

For all nonnegative integers N and k, there is an identity of analytic functions

$$\sum_{m=0}^{\infty}(-1)^{m+N} b_{m+N}^{(k)}(x) \sum_{j=0}^{m}(-1)^j \binom{m}{j}(j+a)^{-s}$$
$$= \frac{\zeta_{N-k}(s-k, a+x)}{(s-1)_k} - \sum_{m=0}^{N-1}(-1)^m b_m^{(k)}(x)\zeta_{N-m}(s,a)$$

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- In this formula, $b_m^{(k)}(x)$ denotes the m-th *Bernoulli polynomial of the second kind* of order k, and $\zeta_N(s,a)$ denotes the Barnes zeta function of order N (with $\zeta_1(s,a)$ denoting the Hurwitz zeta function and $\zeta_1(s,1)$ the Riemann zeta function).

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- The zeta functions $\zeta_N(s,a)$ have simple poles at $s = 1, ..., N$, but the residues on the right hand side sum to zero, to give an entire function.
- The really cool thing is that, for $a, x \in \mathbb{Q}$ and $s \in \mathbb{Z}$, *the exact same series of rational numbers* converges in a p-adic metric, when $x \in \mathbb{Z}_p$ and $|a|_p > 1$, to the entirely analogous combination of p-adic zeta functions.

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$$\left(\frac{t}{e^t - 1}\right)^z e^{xt} = \sum_{n=0}^{\infty} B_n^{(z)}(x) \frac{t^n}{n!}.$$

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- Differences and derivatives satisfy

$$B_n^{(z)}(x+1) - B_n^{(z)}(x) = nB_{n-1}^{(z-1)}(x), \qquad \frac{\partial}{\partial x}B_n^{(z)}(x) = nB_{n-1}^{(z)}(x),$$

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- Either kind of Bernoulli polynomial may be converted to the other by means of

$$n!b_n^{(z)}(x) = B_n^{(n-z+1)}(x+1), \qquad B_n^{(z)}(x) = n!b_n^{(n-z+1)}(x-1).$$

Hurwitz multiple zeta functions

For positive integers r, the Hurwitz zeta function of order r is defined by

$$\zeta_r(s,a) = \sum_{t_1=0}^{\infty} \cdots \sum_{t_r=0}^{\infty} (a + t_1 + \cdots + t_r)^{-s}$$

for $\Re(s) > r$ and $\Re(a) > 0$, and continued meromorphically to $s \in \mathbb{C}$ with simple poles at $s = 1, 2, ..., r$. Note that $\zeta_1(s,a)$ is the Hurwitz zeta function, and $\zeta_0(s,a) = a^{-s}$ by convention.

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- Its values at the negative integers, and residues at its poles, are given by

$$\zeta_r(-k,a) = \frac{(-1)^r k!}{(r+k)!} B_{r+k}^{(r)}(a), \qquad \operatorname*{Res}_{s=k} \zeta_r(s,a) = \frac{(-1)^{r-k} B_{r-k}^{(r)}(a)}{(k-1)!(r-k)!} \quad (k \in \{1, ..., r\}).$$

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- With this definition we have difference and derivative identities

$$\zeta_r(s,a) - \zeta_r(s,a+1) = \zeta_{r-1}(s,a), \qquad \frac{\partial}{\partial a} \zeta_r(s,a) = -s\zeta_r(s+1,a)$$

  for all integers $r \in \mathbb{Z}$.

The weighted Stirling number family of series

For all positive integers k and nonnegative integers N, there is an identity of analytic functions

$$\sum_{m=0}^{\infty}(-1)^{m+N}\frac{s(m+N,k|r)}{(m+N)!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}(j+a)^{-s}$$
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- In this formula $s(m,k|r)$ is the *weighted Stirling number of the first kind*, defined by

$$(1+t)^{-r}(\log(1+t))^k=k!\sum_{m=k}^{\infty}s(m,k|r)\frac{t^m}{m!}\qquad\text{or}\qquad(x)_m=\sum_{k=0}^{m}s(m,k|r)(x+r)^k.$$

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- Again the zeta functions $\zeta_N(s,a)$ have simple poles at $s=1,...,N$, but the residues on the right hand side sum to zero, to give an entire function. The formula also makes good sense p-adically, when $s,x\in\mathbb{Z}_p$ and $|a|_p>1$.

Hyperharmonic special cases

Taking $k = 1$ in this theorem gives the following corollary: For all nonnegative integers r and N we have

$$\sum_{m=0}^{\infty} H_{m+N}^{[r]} \sum_{j=0}^{m} (-1)^j \binom{m}{j} (j+a)^{-s} = s\zeta_N(s+1, a-r) - \sum_{m=1}^{N-1} H_m^{[r]} \zeta_{N-m}(s, a)$$

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- The $r = 0$ case of this formula was the original (1930) formula of Hasse, and the $r = 1$ case was recently given by Blagouchine.

For a prime number p, and $x \in \mathbb{Q}^{\times}$, write $x = p^k(r/s)$ with $(p, r) = (p, s) = (r, s) = 1$ and $k \in \mathbb{Z}$; the integer k is the p-adic valuation of x, denoted $k = \nu_p(x)$; set $\nu_p(0) = +\infty$.

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Define $|x|_p = p^{-\nu_p(x)}$, the p-adic absolute value, for $x \in \mathbb{Q}$.

"High powers of p are small"

p-adic numbers and power functions

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Let $\langle\cdot\rangle$ denote the projection onto the third factor in the internal direct product decomposition $\mathbb{C}_p^\times \cong p^{\mathbb{Q}} \times \mu \times B(1,1^-)$ where μ denotes the group of roots of unity of order not divisible by p.

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We can then define $\langle a\rangle^s = \sum_{n=0}^{\infty} \binom{s}{n}(\langle a\rangle - 1)^n$ for $a \in \mathbb{C}_p^\times$ and $s \in \mathbb{Z}_p$. This is (at least) a C^∞ function of $s \in \mathbb{Z}_p$, and locally analytic as a function of $a \in \mathbb{C}_p^\times$.

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The Iwasawa logarithm $\log_p$ is defined on $\mathbb{C}_p^\times$ by $\log_p a = -\sum_{n=1}^\infty (1 - \langle a\rangle)^n/n$.

The factor $\langle a\rangle/a$, which appears in some of our p-adic formulas, is locally constant and algebraic with p-adic logarithm zero.

p-adic Hurwitz multiple zeta functions

For positive integer orders r, the complex and p-adic multiple zeta functions of order r defined by

$$\zeta_r(s,a) = \sum_{\bar{t}\in\mathbb{Z}_0^r} (a+|\bar{t}|)^{-s}, \qquad \zeta_{p,r}(s,a) = \frac{1}{(s-1)\cdots(s-r)} \int_{\mathbb{Z}_p^r} \frac{(a+|\bar{t}|)^r}{\langle a+|\bar{t}|\rangle^s}\, d\bar{t},$$

where $|\bar{t}| = t_1 + \cdots + t_r$ denotes the "length" of the vector $\bar{t} = (t_1, ..., t_r)$. The p-adic function $\zeta_{p,r}(s,a)$ is (at least) a C^∞ of $s \in \mathbb{Z}_p \setminus \{1, ..., r\}$ when $|a|_p > 1$.

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$$\zeta_r(s,a) = \sum_{\bar{t}\in\mathbb{Z}_0^r} (a+|\bar{t}|)^{-s}, \qquad \zeta_{p,r}(s,a) = \frac{1}{(s-1)\cdots(s-r)} \int_{\mathbb{Z}_p^r} \frac{(a+|\bar{t}|)^r}{\langle a+|\bar{t}|\rangle^s}\, d\bar{t},$$

where $|\bar{t}| = t_1 + \cdots + t_r$ denotes the "length" of the vector $\bar{t} = (t_1, \ldots, t_r)$. The p-adic function $\zeta_{p,r}(s,a)$ is (at least) a C^∞ of $s \in \mathbb{Z}_p \setminus \{1, \ldots, r\}$ when $|a|_p > 1$.

- When $r = 0$ these are just the functions $\zeta_0(s,a) = a^{-s}$ and $\zeta_{p,0}(s,a) = \langle a\rangle^{-s}$.

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- For negative integer orders, we have

$$\zeta_{p,-r}(s,a) = \sum_{j=0}^{r} \binom{r}{j} (-1)^j \langle a+j\rangle^{-s},$$

so that $(-1)^r\zeta_{p,-r}(s,a)$ is the r-th forward difference of the power function $\langle a\rangle^{-s}$ with respect to the a parameter.

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  so that $(-1)^r\zeta_{p,-r}(s,a)$ is the r-th forward difference of the power function $\langle a\rangle^{-s}$ with respect to the a parameter.
- For any integer r, positive or negative, the a-derivative of $\zeta_{p,r}(s,a)$ is an s-shift

$$\frac{\partial}{\partial a}\zeta_{p,r}(s,a) = -s\frac{\langle a\rangle}{a}\zeta_{p,r}(s+1,a).$$

For all nonnegative integers N and k, there is an identity of p-adic analytic functions

$$\sum_{m=0}^{\infty}(-1)^{m+N}b_{m+N}^{(k)}(x)\sum_{j=0}^{m}(-1)^j\binom{m}{j}\langle j+a\rangle^{-s}$$

$$=\left(\frac{a}{\langle a\rangle}\right)^k\frac{\zeta_{p,N-k}(s-k,a+x)}{(s-1)_k}-\sum_{m=0}^{N-1}(-1)^m b_m^{(k)}(x)\zeta_{p,N-m}(s,a)$$

for $s,x\in\mathbb{Z}_p$, provided that $|a|_p>1$.

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- For all positive integers k and nonnegative integers N, there is an identity of p-adic analytic functions

$$\sum_{m=0}^{\infty}(-1)^{m+N}\frac{s(m+N,k|r)}{(m+N)!}\sum_{j=0}^{m}(-1)^{j}\binom{m}{j}\langle j+a\rangle^{-s}$$

$$=\left(\frac{\langle a\rangle}{a}\right)^{k}(-1)^{k}\binom{s+k-1}{k}\zeta_{p,N}(s+k,a-r)-\sum_{m=k}^{N-1}(-1)^{m}\frac{s(m,k|r)}{m!}\zeta_{p,N-m}(s,a)$$

for $s,r\in\mathbb{Z}_p$, provided that $|a|_p>1$.

Taking $N = 1$, $k = 1$, $n = 0$ yields

$$\sum_{m=0}^{\infty} \frac{(-1)^m m! b_{m+1}(x)}{a(a+1)\cdots(a+m)} = \begin{cases} \log(a+x) - \psi(a) \text{ in } \mathbb{R}, & \text{if } a > 0 \text{ and } a + x > 0, \\ \log_p(a+x) - \psi_p(a) \text{ in } \mathbb{C}_p, & \text{if } |a|_p > 1 \text{ and } x \in \mathbb{Z}_p. \end{cases}$$

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- Taking $N=1$, $k=1$, $n=1$, $a=1/2$ gives

$$\sum_{m=0}^{\infty} \frac{(-1)^m 4^{m+1} b_{m+1}(x) O_{m+1}}{(2m+1)\binom{2m}{m}} = \begin{cases} \frac{2}{1+2x} - 3\zeta(2) \text{ in } \mathbb{R}, & \text{if } x > -1/2, \\ \frac{2}{1+2x} - 4\zeta_{2,1}(2, \frac{1}{2}) \text{ in } \mathbb{C}_2, & \text{if } x \in \mathbb{Z}_2, \end{cases}$$

where $O_m := \sum_{j=1}^{m} \frac{1}{2j-1}$ is the m-th "odd harmonic" number.

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- Taking $N = 1,\ k = 1,\ n = 0,\ x = -1,\ a = 3/2$ gives

$$\sum_{m=0}^{\infty} \frac{(-1)^m 4^{m+1} B_{m+1}^{(m+1)}}{(m+1)(2m+3)(m+1)!\binom{2m+2}{m+1}} = \begin{cases} -\log 2 - \psi(3/2) & \text{in } \mathbb{R}, \\ -\log_2 2 - \psi_2(3/2) & \text{in } \mathbb{Q}_2. \end{cases}$$

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p-adic Stirling series examples at positive integers

Taking $N = 1$, $k = 1$, $n = 0$, $a = r + \frac{1}{2}$ yields

$$\sum_{m=0}^{\infty} \frac{4^{m+1}\binom{2r}{r} H_{m+1}^{[r]}}{(2r+2m+1)\binom{2m+2r}{m+r}\binom{m+r}{r}} = \begin{cases} 6\zeta(2) = \pi^2 & \text{in } \mathbb{R}, \\ 8\zeta_{2,1}(2, \frac{1}{2}) = 0 & \text{in } \mathbb{Q}_2. \end{cases}$$

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$$\sum_{m=0}^{\infty} \frac{4^{m+1}\binom{2r}{r} H_{m+1}^{[r]}(O_{r+m+1}-O_r)}{(2r+2m+1)\binom{2m+2r}{m+r}\binom{m+r}{r}} = \begin{cases} 7\zeta(3) & \text{in } \mathbb{R}, \\ 16\zeta_{2,1}(3,\frac{1}{2}) & \text{in } \mathbb{Q}_2. \end{cases}$$

where $O_m := \sum_{j=1}^{m} \frac{1}{2j-1}$ is the m-th "odd harmonic" number.

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- Kurokawa, Tanaka, and others realized that you needed a regularization of the Connes-Consani zeta function to accomodate varieties with counting functions which don't vanish at $q = 1$.

Absolute zeta functions

Let $N(x)$ be a suitable "counting function" defined on $(0, \infty)$ satisfying an automorphic relation of the form

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where D is an integer and $C = \pm 1$.

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- The connection to my work on zeta expansions stems from the relation of $\zeta_{\mathbb{G}_m^n/\mathbb{F}_1}$ to the negative-order zeta functions $\zeta_{-n}(s, a)$.

Absolute limit formulas for Hurwitz zeta functions

Denote by $\gamma_k(a)$ the constant term in the Laurent expansion of the order k zeta function $\zeta_k(s,a)$ at its rightmost pole, that is,

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$$\gamma_k(a-r) + \frac{H_{k-1}}{(k-1)!} = k \sum_{n=0}^{\infty} \frac{(-1)^n s(n+k,k|r)}{(n+k)!} \log \zeta_{\mathbb{G}_m^n/\mathbb{F}_1}(a+n).$$

  In particular, for every nonnegative integer r we have

$$\gamma(a) = \sum_{n=0}^{\infty} H_{n+1}^{[r]} \log \zeta_{\mathbb{G}_m^n/\mathbb{F}_1}(a+r+n),$$

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- This was observed in the case $r=0$ by Kurokawa and Tanaka in 2018. In the case $r=0$, $a=1$, $k=1$, it reduces to a 1776 result of Euler.

More absolute limit formulas for Hurwitz zeta functions

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- These series differ in spirit from those of Kurokawa and Tanaka, as they involve the absolute Hurwitz zeta function for $\mathbb{G}_m^n$ which occurs in the regularization process for the absolute Hasse zeta function.

More absolute limit formulas for Hurwitz zeta functions

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$$\gamma = \frac{\zeta(2)}{2} - \sum_{n=0}^{\infty} (-1)^n b_{n+2}^{(2)} Z_{\mathbb{G}_m^n/\mathbb{F}_1}(2, n+1).$$

p-adic absolute limit formulas for Hurwitz zeta functions

Denote by $\gamma_{p,k}(a)$ the constant term in the Laurent expansion of the order k p-adic zeta function $\zeta_{p,k}(s, a)$ at its rightmost pole, that is,

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  In particular, for every nonnegative integer r we have

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- Recall that the heuristic of $\mathbb{F}_1$ is that one constructs a variety over actual finite fields $\mathbb{F}_q$ and then hypocritically takes the "limit" of the counting function, and zeta function, as $q \to 1$. But in the p-adic world there is no hypocrisy because 1 actually is a p-adic limit point of the set of prime powers!

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- I suspect that the fact that these series expansions are also valid p-adically is giving additional algebraic information, but I don't have a precise statement yet.

Thank You!