Factoradic Happy Numbers

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Motivation

Richard Guy's *Unsolved Problems in Number Theory* (1994)

The *happy function* is $S : \mathbb{Z}^+ \to \mathbb{Z}^+$ sending n to the sum of the squares of its decimal digits.

Example: $S(51) = 5^2 + 1^2 = 26$

A *happy number* is a number which arrives at 1 after repeated iterations of the happy function.

Example: $S^{10}(51) = 4 \implies$ 51 is unhappy

$4 \to 16 \to 37 \to 58 \to 89 \to 145 \to 42 \to 20 \to 4$

Are there sequences of happy numbers of arbitrary length?

2000 El-Sedy and Siksek:

Let $n = \sum_{i=0}^{k} a_i \cdot 10^i$. Then $S(n) = \sum_{i=0}^{k} a_i^2 \implies$ YES!

2001 Grundman and Teeple: generalized happy numbers

For $n = \sum_{i=0}^{k} a_i \cdot b^i$. Define $S_{e,b}(n) = \sum_{i=0}^{k} a_i^e$

2007 Grundman and Teeple: for $e \geq 2$, and some b

Let $n = \sum_{i=0}^{k} a_i \cdot b^i$. Then $S_{e,b}(n) = \sum_{i=0}^{k} a_i^e \implies$ YES!

2019 Carlson, G, Harris: factoradic base?

More Definitions: Factoradic Happy Things

Let $n \in \mathbb{Z}^+$. Write $n = \sum_{i=0}^{k} a_i \cdot i!$ with $a_k \neq 0$ and $0 \leq a_i \leq i$.

Example: $51 = 2 \cdot 4! + 0 \cdot 3! + 1 \cdot 2! + 1 \cdot 1! + 0 \cdot 0!$

Define the *generalized factoradic happy function* as

$S_{e,!} : \mathbb{Z}^+ \to \mathbb{Z}^+$ by $S_{e,!}(n) = (n) = \sum_{i=1}^{k} a_i^e$.

An *e-power factoradic happy number* is a number which arrives at 1 after repeated iterations of the generalized factoradic happy function.

Example: $S_{2,!}^2(51) = 1 \implies$ 51 is happy!

And More Definitions

For $n = \sum_{i=1}^{k} a_i \cdot i!$ with $a_k \neq 0$ and $0 \leq a_i \leq i$ for $1 \leq i \leq k$.

We say $p \in \mathbb{Z}^+$ is an *e-power factoradic fixed point* if $S_{e,!}(p) = p$.

Example: $S_{2,!}(5) = 5$

We call $n \in \mathbb{Z}^+$ an *e-power factoradic p-happy number* if $\exists \ell \geq 1$ s.t. $S_{e,!}^{\ell}(n) = p$.

Example: $S_{2,!}^{3}(2021) = 5$

e-Power Factoradic Fixed Points

e	M_e	e-power factoradic fixed points	Cycles
1	5	1	None
2	23	1, 4, 5	None
3	119	1, 16, 17	None
4	5039	1, 658, 659	None
5	40319	1, 34, 35, 308, 309, 1058, 1059	(3401,2114)
6	362879	1, 8258, 8259	(731, 67, 794)

Main Result

Theorem (Carlson, G, Harris, 2019)

For $e \in \{1, 2, 3, 4\}$ and for any e-power factoradic fixed point p of $S_{e,!}$, there exists arbitrarily long sequences of e-power factoradic p-happy numbers.

YES!

Idea of Proof in Cases

Case $e = 1$ is straightforward.

For cases $e \in \{2, 3, 4\}$:

Lemma

Let $e \in \{2, 3, 4\}$. If $j_e \in \mathbb{Z}^+$ is smallest s.t. $j_e! > j_e^{e-1}$, then $\forall k \geq j_e$,

$$k! > k^{e-1} \text{ and } (k+1)! - (k+1)^{e-1} \geq k! - k^{e-1}.$$

Define $M_e = \displaystyle\sum_{i=1}^{j_e} i \cdot i!$.

Theorem

Let $e \in \{2, 3, 4\}$ and $j_e \in \mathbb{Z}^+$ smallest s.t. $j_e! > j_e^{e-1}$. If $n \in \mathbb{Z}^+$ with $n > M_e$, then $n > S_{e,!}(n)$.

Idea of Proof

e	M_e	e-power factoradic fixed points	Cycles
1	5	1	None
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Future Work

- What happens with $e = 5, 6$ when there are cycles?
- What can we say about the density of each factoradic p-happy number?

e	Proportions of e-power factoradic fixed points of $S_{e,!}$ in the interval $I = [1, 10!]$		
2	$P_{2,1}(I) = \frac{2220945}{10!} = 0.612,$	$P_{2,4}(I) = \frac{244026}{10!} = 0.067,$	$P_{2,5}(I) = \frac{1163828}{10!} = 0.321$
3	$P_{3,1}(I) = \frac{3421678}{10!} = 0.943,$	$P_{3,16}(I) = \frac{31856}{10!} = 0.009,$	$P_{3,17}(I) = \frac{175265}{10!} = 0.048$
4	$P_{4,1}(I) = \frac{3556797}{10!} = 0.980,$	$P_{4,658}(I) = \frac{29574}{10!} = 0.008,$	$P_{4,659}(I) = \frac{42428}{10!} = 0.012$
5	$P_{5,1}(I) = \frac{179930}{10!} = 0.049,$	$P_{5,34}(I) = \frac{1545589}{10!} = 0.426,$	$P_{5,35}(I) = \frac{38188}{10!} = 0.0105,$
	$P_{5,308}(I) = \frac{120298}{10!} = 0.033,$	$P_{5,309}(I) = \frac{200223}{10!} = 0.055,$	
	$P_{5,1058}(I) = \frac{357868}{10!} = 0.0986,$	$P_{5,1059}(I) = \frac{139821}{10!} = 0.0385$	

Values for $P_{e,p}(I)$ for $e \in \{2, 3, 4, 5\}$ and $I = [1, 10!]$.

Thank you!