Binary Curves of Fixed Genus and Gonality with Many Points

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Work

This is ongoing joint work with Xander Faber.

manypoints.org

- For given (small) finite field and (small) genus, what is the maximum number of points a smooth, projective curve can have?

manypoints.org

- For given (small) finite field and (small) genus, what is the maximum number of points a smooth, projective curve can have?
- There is a fairly extensive database at `manypoints.org`.
- For $\mathbb{F}_2$ it looks like:

g	$N_2(g)$
0	3
1	5
2	6
3	7
4	8
5	9

Gonality

- The **gonality** γ of a curve X over a field k is the minimum degree of a k-morphism $X \to \mathbb{P}^1$.

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- Gonality 1 curves are isomorphic to $\mathbb{P}^1$, so coincide with genus 0 curves.
- Gonality 2 curves are **hyperelliptic**, and include elliptic curves (genus 1 and up).
- Gonality 3 curves are known as **trigonal** curves.

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  or

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  or
- It's fun and interesting.

Let's start a table for binary curves

- Proposition: If $g = 0$, then $\gamma = 1$. If $g = 1$, then $\gamma = 2$. If $g \geq 2$, then $\gamma \leq 2g - 2$.

γ \ g	0	1	2	3	4	5
1	3					
2		5	6	?	?	?
3				?	?	?
4				?	?	?
5					?	?
6					?	?
7						?
8						?

Hyperelliptic curves

- Proposition: The number of points on a binary curve of gonality γ is $\leq 3\gamma$.
- Proposition: For each genus $g \geq 2$, there exists a hyperelliptic curve over $\mathbb{F}_2$ with 6 rational points.
- Proof: Look at $y^2 + [1 + x^g(x+1)]\ y = [x(x+1)]^{g-\delta} = 0$, where $\delta = g \pmod 2$.

γ \ g	0	1	2	3	4	5
1	3					
2		5	6	6	6	6
3				?	?	?
4				?	?	?
5					?	?
6					?	?
7						?
8						?

Genus 3, Gonality 3

- Proposition: For a curve with rational points, $\gamma \leq g$.

g / γ	0	1	2	3	4	5
1	3					
2		5	6	6	6	6
3				7	?	?
4				?	?	?
5					?	?
6					?	?
7						?
8						?

Genus 3, Gonality 4

- A genus-3 curve with rational points must have gonality ≤ 3.
- $(x^2 + xz)^2 + (x^2 + xz)(y^2 + yz) + (y^2 + yz)^2 + z^4 = 0$ has gonality 4.
- Demonstrate lack of degree-3 morphism by looking at $\mathbb{F}_4$ points.

γ \ g	0	1	2	3	4	5
1	3					
2		5	6	6	6	6
3				7	?	?
4				0	?	?
5					?	?
6					?	?
7						?
8						?

Fun facts about genus 4 non-hyperelliptic curves

- Non-hyperelliptic curves of genus 4 can be embedded as the intersection of a quadric surface and a cubic surface.
- If the quadric surface is $xy + zw = 0$ or $xy + z^2 = 0$, the curve is trigonal.
- If the quadric surface is $xy + z^2 + wz + w^2 = 0$, the curve is not.

Genus 4, Gonality 3

- Consider the curve:

$$\begin{aligned} xy + zw &= 0 \\ xy^2 + y^3 + x^2z + y^2z + xz^2 + x^2w + y^2w + xw^2 &= 0. \end{aligned}$$

- It has 8 points and is trigonal.

Genus 4, Gonality 4

- The surface $xy + z^2 + wz + w^2 = 0$ has only 5 rational points.
- Consider the curve:

$$\begin{aligned} xy + z^2 + zw + w^2 &= 0 \\ xy^2 + x^2z + y^2z + yz^2 + x^2w + z^2w &= 0. \end{aligned}$$

- It has 5 points.

Genus 4, Pointless Curves

- If a genus 4 curve has gonality 5 or 6, it must be pointless.
- Consider the curve:

$$\begin{aligned} xy + z^2 + zw + w^2 &= 0 \\ x^3 + y^3 + z^3 + y^2w + xzw &= 0. \end{aligned}$$

- Not gonality 4; look at $\mathbb{F}_{16}$ points.
- Write down degree-5 morphism.
- Do this for all pointless curves to rule out gonality 6. (Computations!)

Updated Table

γ \ g	0	1	2	3	4	5
1	3					
2		5	6	6	6	6
3				7	8	?
4				0	5	?
5					0	?
6						?
7						?
8						?

Genus 5, Gonality 3 and 4

- Trigonal curves of genus 5 are birationally equivalent to plane quintics with a multiplicity-2 singularity.
- This gives an upper bound of $2^2 + 2 + 1 + 1$ (based on the size of $\mathbb{P}^2$).
- We achieve this bound with

$$xyz^3 + x^3z^2 + y^3z^2 + x^4z + xy^3z + y^4z + x^4y + x^2y^3 = 0.$$

- The genus-5 curves on `manypoints.org` with 9 points have degree-4 morphisms and thus gonality 4.

Genus 5, Gonality 5 and up

- Non-hyperelliptic, non-trigonal genus-5 curves are intersections of three quadric surfaces.
- Can exhaust over pointless genus-5 curves to rule out gonality above 5.
- Can exhaust over all genus-5 curves by looking at possible divisors to show that the maximum number of points on a gonality-5 curve is 3.

Current Table

γ \ g	0	1	2	3	4	5
1	3					
2		5	6	6	6	6
3				7	8	8
4				0	5	9
5					0	3
6						
7						
8						

Future Work

- Refine gonality computation code and algorithm.
- Ternary curves.
- `gonality.org`