Tamely Ramified Covers of the Projective Line and Markoff Triples

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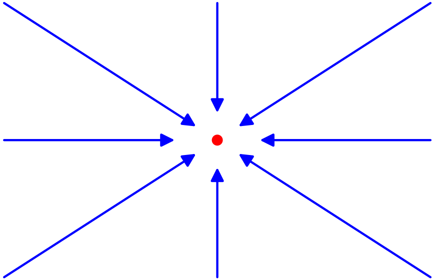
Covers of the Plane

Let $X = \mathbf{A}^1 = \mathbf{P}^1 - \{\infty\}$, p a prime.

- Over $\mathbf{C}$, it's simply connected. $\pi_1(X_{\mathbf{C}}) = 0$



- Over $\overline{\mathbf{F}}_p$, $\pi_1^{\text{ét}}(X_{\overline{\mathbf{F}}_p}) \neq 0$. There are Artin-Schreier curves like $y^p - y = x^d$; the map $(x, y) \mapsto x$ is a $\mathbf{Z}/p\mathbf{Z}$-Galois cover.

Several Perspectives

Let X be a smooth projective curve over $k = \overline{\mathbf{F}}_p$, and B a finite set of points of X.
By default, all curves and covers are connected.

The following are equivalent:

- Finite index open subgroups of $\pi_1^{\text{ét}}(X - B)$;
- Finite étale covers of $X - B$;
- Finite branched covers of X ramified only over B;
- Finite extensions of $k(X)$ ramified only over B.

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The monodromy group of the cover is the Galois group of the normal closure of the field extension.

What is Known

X curve of genus g, $\#B = r$.

- Over $\mathbf{C}$, the fundamental group of a Riemann surface of genus g with r points removed is $\Gamma_{g,r}$, a free on $2g + r - 1$ generators. ("Topological Fundamental Group")
- Let $p(G)$ be the subgroup of G generated by the p-Sylow subgroups.

Theorem (Harbater, Raynaud)

Over $\overline{\mathbf{F}}_p$, if $r > 0$ then G is the Galois group of a finite étale cover of $X - B$ if and only if $G/p(G)$ is generated by $2g + r - 1$ elements.

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- The finite quotients of $\pi_1^{\text{ét}}(X - B)$ do not determine this pro-finite group.

Tame Ramification

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A potentially manageable question:

Question

What are the tamely ramified covers of $X - B$?

Tamely Ramified Covers

Specialize to $X = \mathbf{P}^1$, $B = \{0, 1, \infty\}$.

- There is a G-Galois cover of $\mathbf{P}^1 - \{0, 1, \infty\}$ in characteristic zero if and only if G is generated by two elements.
- Which of these show up as Galois groups of tame covers in characteristic p?

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- Strategy: take a G-Galois cover defined in characteristic zero, reduce it modulo p.

A Criterion for (Potentially) Good Reduction

Let G be a finite group generated by two elements.

Theorem (Raynuad, Obus)

Suppose G has **cyclic** *p-Sylow subgroup. Let $K_0 = \mathrm{Frac}(W(k))$, and K/K_0 be a finite extension of degree $e(K)$, where $e(K)$ is less than the number of conjugacy classes of order p in G. If $\pi : Y \to \mathbf{P}^1 - \{0, 1, \infty\}$ is a G-Galois cover defined over K, then π has potentially good reduction.*

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Example

Gives tamely ramified $\mathrm{PGL}_m(\mathbf{F}_q)$-covers in characteristic p for well-chosen m and q.

New Tamely Ramified Covers

Theorem

Fix a prime p, and let $k = \overline{\mathbf{F}}_p$. For infinitely many n, there exists a curve C over k and a branched Galois cover $\pi : C \to \mathbf{P}^1_k$ tamely ramified over three points and unramified elsewhere, whose Galois group isomorphic to the symmetric group S_n (and likewise for the alternating group A_n).

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Strategy: construct cover from a map between moduli spaces

Moduli of Elliptic Curves with G-structure

Fix a finite group G generated by two elements; assume for simplicity that $Z(G) = 1$.

Definition

Let $\mathcal{M}(G)$ be the moduli space of elliptic curves with G-structure: an elliptic curve E together with a Galois cover of $E - \{\mathcal{O}\}$ with Galois group G.

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- If G is Abelian, a different formulation recovers modular curves with familiar level structures: $X(N)$ corresponds to $G = (\mathbf{Z}/n\mathbf{Z})^2$.
- This is naturally defined over $\mathbf{Z}[\frac{1}{|G|}]$.

Moduli of Elliptic Curves with G-structure

- Forgetful map $p : \mathcal{M}(G) \to \mathcal{M}(1)$ is finite étale.
- Over $\mathbf{C}$: get branched cover $M(G)_{\mathbf{C}} \to \mathbf{P}^1_{\mathbf{C}}$ ramified over 0, 1728, ∞. It's easy to understand ramification.

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- Interpretation of fibers over $\mathbf{C}$: $\mathrm{Surj}(F_2, G)/\,\mathrm{Inn}(G)$ with monodromy action of $\mathrm{SL}_2(\mathbf{Z})$
- Experiments show: if $G = \mathrm{PSL}_2(\mathbf{F}_\ell)$, then there is a natural large orbit of size n where the monodromy action is S_n or A_n

Markoff Triples

Let X be the surface $x^2 + y^2 + z^2 - 3xyz = 0$. 3

- $X(\mathbf{Z})$: Markoff triples
- Markoff group Γ: permute coordinates, $(x, y, z) \mapsto (3yz - x, y, z)$ . . .

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- $X(\mathbf{Z})$: Markoff triples
- Markoff group Γ: permute coordinates, $(x, y, z) \mapsto (3yz - x, y, z)$ ...
- $X^*(\mathbf{F}_\ell) = X(\mathbf{F}_\ell) - \{(0,0,0)\}$
- Experimentally: $X^*(\mathbf{F}_\ell)$ is a single Γ-orbit, action factors through symmetric or alternating group

Proving Things

The following are closely linked:

- Monodromy on fibers of $M(\mathrm{PSL}_2(\mathbf{F}_\ell))_{\mathbf{C}} \to \mathbf{P}^1_{\mathbf{C}}$;
- Action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathrm{Surj}(F_2, \mathrm{PSL}_2(\mathbf{F}_\ell))/\,\mathrm{Inn}(\mathrm{PSL}_2(\mathbf{F}_\ell))$;
- Action of Γ on $X^*(\mathbf{F}_\ell)$.

Given $\phi : F_2 = \langle a, b\rangle \to \mathrm{PSL}_2(\mathbf{F}_\ell)$ in the "preferred component",

$$(\mathrm{tr}\phi(a), \mathrm{tr}\phi(b), \mathrm{tr}\phi(ab)) \in X^*(\mathbf{F}_\ell)$$

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- Bourgain, Gamburd, and Sarnak show: there is always a large Γ-orbit on $X^*(\mathbf{F}_\ell)$.
- Can adapt work of Meiri and Puder to see that: for infinitely many ℓ, Γ acts as symmetric or alternating group on this orbit.

Thank you.